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Berezinskii–Kosterlitz–Thouless transitions in the six-state clock model

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Abstract

A classical 2D clock model is known to have a critical phase with Berezinskii–Kosterlitz–Thouless (BKT) transitions. These transitions have logarithmic corrections which make numerical analysis difficult. In order to resolve this difficulty, one of the authors has proposed a method called 'level spectroscopy', which is based on the conformal field theory. We extend this method to the multi-degenerated case. As an example, we study the classical 2D six-clock model which can be mapped to the quantum self-dual 1D six-clock model. Additionally, we confirm that the self-dual point has a precise numerical agreement with the analytical result, and we argue the degeneracy of the excitation states at the self-dual point from the effective field theoretical point of view.

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1. Introduction

Although low-dimensional spin systems with a continuous rotational symmetry, such as the 2D XY spin model, do not have a symmetry breaking phase at finite temperatures, it does not mean the absence of phase transitions. Actually Berezinskii [1] and Kosterlitz and Thouless [2] have pointed out a phase transition induced by vortex–antivortex excitation in the 2D XY model, which is known as the Berezinskii–Kosterlitz–Thouless (BKT) transition.

The classical 2D clock model, which has a discrete rotational symmetry, is also known to have a BKT critical region. José *et al* [3] have studied the 2D classical p-state clock model by the renormalization group analysis and pointed out the presence of the intermediate BKT critical phase for sufficiently large p. Elitzur *et al* [4] have found strong evidence for existence of a massless region, that is a BKT critical region, for $p \geqslant 5$. To estimate the mass gap, they used a correlation inequality and duality. Henceforth, duality means the exchange of a high-temperature region and low-temperature region. They also mentioned that the duality in a finite system requires a simple relation between a charge symmetry and boundary conditions.

We map the 2D p-clock model onto its quantum 1D p-clock model and analyse the latter. This is based on the fact that the $(1+d)$-dimensional classical statistical mechanics can be mapped onto the d-dimensional quantum statistical mechanics. Here, we should be careful that the 2D classical clock model does not have self-duality *explicitly*, but its quantized Hamiltonian has self-duality. It may seem strange, but thinking of the universality class, that is, the Z_p Villain model, which is known to have self-duality, and the 2D classical p-clock model belong to the same universality class, the self-duality of the quantum clock model is acceptable. Such an argument has been done in [4]. Also the corresponding field theoretical model (the dual sine-Gordon model) has self-duality. We will see an excellent numerical result about the self-duality.

Generally, the BKT transition points were difficult to determine numerically because of the logarithmic corrections and the anomalous divergence of the correlation length. To resolve these difficulties, the level spectroscopy method [5, 6] has been proposed. Near a BKT transition point, some scaling dimensions change from relevant to irrelevant or vice versa. Choosing some appropriate scaling dimensions, we can determine the BKT transition point by the crossing point of these scaling dimensions. In addition, we can eliminate logarithmic corrections accompanying the BKT transition. Therefore we can accurately determine the numerical BKT transition points.

So far the level spectroscopy has been applied to the three cases. The first case is between the BKT critical phase and the phase of the non-degenerate ground state [6, 7]. The second is between the BKT critical phase and the phase of twofold degenerate ground state [5]. The third is between the BKT critical phase and the multi-degenerated ground state phase. Recently, Tonegawa *et al* [8] studied the $1/3$ plateau problem of the $S=\frac{1}{2}$ antiferromagnetic XXZ chain with the Z_3 symmetry breaking and Otsuka *et al* [9] studied the 2D AF three-state Potts model with the Z_6 symmetry breaking case.

In this paper, we apply the level spectroscopy of the multi-degenerated case [8, 9] to the 1D quantum six-clock model with duality and use the duality relation to check the level spectroscopy results between the BKT critical phase and the multi-degenerated ground state from another point of view than former studies. In addition, we discuss the degeneracy at the self-dual point from the field theoretical point of view. The self-dual point of this model is trivial but it will be important in the 2D AF three-state Potts model [9] which has no explicit self-duality.

2. Theory

2.1. Discrete model

In this subsection, we review the 1D quantum p-state clock model with a nearest-neighbour interaction as a Hamiltonian limit of a 2D classical p-state clock model on a square lattice. The 2D classical Hamiltonian is

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \{J_s[\cos(\Theta_{i,j} - \Theta_{i+1,j}) - 1] + J_\tau[\cos(\Theta_{i,j} - \Theta_{i,j+1})]\}, \tag{1}$$

where $\Theta_{i,j} = (2\pi r/p), r = 0, 1, \ldots, p-1$, is the clock spin variable. When we define λ to satisfy the equation $\lambda\beta J_s = \mathrm{e}^{-2\beta J_\tau[\cos(2\pi/p)-1]}$ and take the Hamiltonian limit $\beta J_s \to 0$, $\beta J_\tau \to \infty$ with λ fixed [10], we obtain the 1D quantum Hamiltonian [4, 10]

$$H = -2\sum_{n=1}^{N} \{\lambda \cos(\hat{\Theta}_n - \hat{\Theta}_{n+1}) + \cos \hat{p}_n x\}, \tag{2}$$

where λ is the transverse field of the system. It can be interpreted as the inverse of the temperature. See [11, 17] for a more detailed derivation.

When we define the operators σ_n, Γ_n as $\sigma_n = \exp \mathrm{i}\hat{\Theta}$, $\Gamma_n = \exp \mathrm{i}\hat{p}$, we obtain another representation of the 1D quantum Hamiltonian (2) as

$$H = -\sum_{n=1}^{L} \left\{ \lambda \left(\sigma_n \sigma_{n+1}^{+} + \sigma_n^{+} \sigma_{n+1} \right) + \Gamma_n + \Gamma_n^{+} \right\}, \tag{3}$$

where

$$\sigma_n = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{p-1} \end{pmatrix}, \qquad \Gamma_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}, \qquad \omega = \exp(2\pi \mathrm{i}/p)$$

in the σ_n diagonal representation. These operators satisfy $(\sigma_n)^p = (\Gamma_n)^p = 1$.

This model has a Z_p symmetry. The corresponding Z_p charge operator is defined as

$$U_Q = \prod_{n=1}^{L} \Gamma_n, \tag{4}$$

which commutes with the Hamiltonian (3). U_Q has eigenvalues ω^Q, $Q = 0, 1, \ldots, p-1$. Because of the conservation, U_Q splits eigenspaces of H into the charge sectors whose corresponding eigenspaces are labelled by $Q = 0, 1, \cdots, p-1$, i.e. H is able to be block-diagonalized.

Also we can block-diagonalize it with the toroidal boundary conditions

$$\sigma_{N+1} = \exp\left(\frac{2\pi \mathrm{i}}{p} \tilde{Q} \right) \sigma_1, \qquad \tilde{Q} = 0, 1, \ldots, p-1. \tag{5}$$

$\tilde{Q} = 0$ is the periodic boundary condition and $\tilde{Q} = p/2$ (p even) is the twisted boundary condition. Hereafter we will assume that p is even. Implementing the twisted boundary condition to the system corresponds to adding a half-charge to the sine-Gordon model [7, 12] which is the continuum limit of the discrete model.

There exists a unitary operator such that σ_n is transformed into Γ_n^{+} and also Γ_n is transformed into σ_n. For example we can take its matrix elements as

$$U_{i,j} = \frac{1}{\sqrt{p}} \omega^{(1-i)(1-j)}. \tag{6}$$

This operator U gives the relation

$$U \sigma_n U^{+} = \Gamma_n^{+}, \tag{7}$$

$$U \Gamma_n U^{+} = \sigma_n. \tag{8}$$

By this relation, we get another representation of the 1D quantum Hamiltonian as

$$H' = -\sum_{n=1}^{L} \left\{ \lambda \left(\Gamma_n \Gamma_{n+1}^{+} + \Gamma_n^{+} \Gamma_{n+1} \right) + \sigma_n + \sigma_n^{+} \right\}. \tag{9}$$

In this case, note that the Z_p charge operator should be

$$U_Q' = U U_Q U^{+} = \prod_{n=1}^{L} \sigma_n. \tag{10}$$

The form (9), (10) is more useful for numerical calculation than the previous form (3), when we choose the σ_n diagonal representation.

There is another non-local transformation between σ and Γ:

$$\begin{aligned}
&\sigma_n \to \tilde{\sigma}_n = \Gamma_{n+1}\Gamma_n^+ \qquad \text{for} \quad n < N \\
&\sigma_N \to \tilde{\sigma}_N = \Gamma_N^+ \\
&\Gamma_n \to \tilde{\Gamma}_n = \prod_{l=1}^{n} \sigma_l.
\end{aligned} \tag{11}$$

Applying this transformation to the Hamiltonian (9) and with the help of

$$\tilde{\Gamma}_n \tilde{\Gamma}_{n+1}^+ = \prod_{l=1}^{n} \sigma_l \prod_{l=1}^{n+1} \sigma_l^+ = \sigma_{n+1}^+, \tag{12}$$

we get the dual form of (9) as

$$-\sum_{n=1}^{L} \left[\lambda\left(\sigma_n^+ + \sigma_n\right) + \Gamma_n \Gamma_{n+1}^+ + \Gamma_n^+ \Gamma_{n+1}\right]. \tag{13}$$

Therefore the 1D p-state quantum clock model satisfies the following duality relation:

$$H(\lambda) = \lambda H(1/\lambda). \tag{14}$$

The self-dual point is $\lambda_c = 1$.

The charge conjugation operator is defined as

$$C = \prod_{n=1}^{L} c_n, \qquad c_n = \begin{pmatrix} 1 & & & & 0 \\ & 0 & & 0 & 1 \\ & & \ddots & 1 & \\ & 0 & 1 & \ddots & \\ 0 & 1 & & & 0 \end{pmatrix}, \tag{15}$$

which has eigenvalues ± 1 and transforms σ_n and Γ_n as

$$c_n \sigma_n c_n^+ = \sigma_n^+ \tag{16}$$

$$c_n \Gamma_n c_n^+ = \Gamma_n^+. \tag{17}$$

By this relation, it is easy to see that the Hamiltonians (3) and (9) commute with C.

Under the charge conjugation, Z_p charge operator U_Q satisfies

$$C U_Q C = U_Q^+. \tag{18}$$

So the eigenvalues ω^Q of U_Q become ω^{-Q}, that is, generally the eigenstates of U_Q are not those of C. But the eigenstates of $Q = 0, p/2$ are invariant under this transformation; therefore these eigenstates have eigenvalues $C = \pm 1$. We will use this fact to classify the eigenstates.

2.2. Effective model

As an effective theory of a 1D quantum spin system, the sine-Gordon model has been studied. Here, we will explain the Z_p dual sine-Gordon model [13]

$$\mathcal{L} = \frac{1}{2\pi K}(\nabla\phi)^2 + \frac{y_\phi}{2\pi\alpha^2}\cos\sqrt{2}\phi + \frac{y_\theta}{2\pi\alpha^2}\cos p\sqrt{2}\theta, \tag{19}$$

where K is also the inverse of temperature as λ, α is an ultraviolet cutoff and ϕ and θ are periodic:

$$\phi = \phi + \frac{2\pi}{\sqrt{2}}, \tag{20}$$

$$\theta = \theta + \frac{2\pi}{\sqrt{2}}. \tag{21}$$

The field θ is dual to ϕ and they satisfy the relation

$$\partial_x \phi = -\partial_y (K\theta), \qquad \partial_y \phi = \partial_x (K\theta). \tag{22}$$

This model is the effective model for the Hamiltonians (3) and (9).

Vertex operators are defined as

$$O_{m,n} = \exp(\mathrm{i}m\sqrt{2}\theta)\exp(\mathrm{i}n\sqrt{2}\phi). \tag{23}$$

When $y_\phi = y_\theta = 0$, their scaling dimensions are

$$x_{m,n}(K) = \frac{1}{2}\left(n^2 K + \frac{m^2}{K}\right). \tag{24}$$

In a high-temperature region (K small) the second term of (19) is more relevant than the third term. So the renormalization group behaviour in the high-temperature region can be studied through the Lagrangian

$$\mathcal{L}_{\mathrm{high}} = \frac{1}{2\pi K}(\nabla\phi)^2 + \frac{y_\phi}{2\pi\alpha^2}\cos\sqrt{2}\phi. \tag{25}$$

When $K = 4$ this sine-Gordon model shows the BKT transition, and this is studied by the level spectroscopy [7]. On the other hand, in the low-temperature region the second term of (19) is more irrelevant than the third term. Then we deal with the sine-Gordon Lagrangian

$$\mathcal{L}_{\mathrm{low}} = \frac{K}{2\pi}(\nabla\theta)^2 + \frac{y_\theta}{2\pi\alpha^2}\cos p\sqrt{2}\theta \tag{26}$$

with the Z_p symmetry breaking term. Here we got the first term by substituting (19) into (22).

Under the change of the cutoff $\alpha \to e^l\alpha$ and parameterizing $K/4 = 1 + y_0/2$, the renormalization group equations for (25) are

$$\frac{\mathrm{d}y_0(l)}{\mathrm{d}l} = -y_\phi^2(l), \qquad \frac{\mathrm{d}y_\phi(l)}{\mathrm{d}l} = -y_\phi(l)y_0(l), \tag{27}$$

where l is related to $e^l = L$ for the finite system size. Here, $y_\phi = 0$ is a fixed points line, y_ϕ is irrelevant for $y_0 > 0$ and relevant for $y_0 < 0$. In the region $|y_\phi| < y_0$, the renormalization group trajectories flow to the line of $y_\phi = 0$ for $l \to \pm\infty$. On the other hand, for $|y_\phi| > y_0$, the renormalization group trajectories flow to $y_\phi = \pm\infty$. The line of $y_\phi(l) = \pm y_0(l)$ is the BKT critical line, and on this line, $y_0(l)$ behaves as $1/l$ $(= 1/\ln L)$.

Similarly, for the case (26), by parameterizing $K^{-1}p^2/4 = 1 + y_0/2$ and using y_θ instead of y_ϕ, we get the renormalization equations

$$\frac{\mathrm{d}y_0(l)}{\mathrm{d}l} = -y_\theta^2(l), \qquad \frac{\mathrm{d}y_\theta(l)}{\mathrm{d}l} = -y_\theta(l)y_0(l). \tag{28}$$

In this case, we can make the same argument as before, replacing $y_\phi(l)$ with $y_\theta(l)$ in equations (27).

In the high-temperature limit $\lambda \to 0$ (or $K \to 0$) this model has a non-degenerated ground state. The transition point between the non-degenerated ground state phase and the BKT critical phase can be determined by the $K = 4$ version of the level spectroscopy. In this

Table 1. Operator content of the $K = p^2/4$ sine-Gordon model with Z_p symmetry breaking term. TBC is short for the twisted boundary condition. Others are under periodic boundary condition. t is defined by $y_\theta = \pm y_0(1+t)$. Marginal-like and $\cos(\sqrt{2}p\theta)$-like hybridize [6] because they have the same quantum numbers.

Field	Quantum number Q	Quantum number C	BC	Scaling dimension $x_{m,n}$	Renormalized scaling dimension
$\exp(im\sqrt{2}\theta)$	$m \neq p/2, p$		PBC	$x_{m,0}$	$\frac{2m^2}{p^2}\left(1+\frac{1}{2}y_0(l)\right)$
$\cos(\sqrt{2}p\theta/2)$	$p/2$	1	PBC	$x_{p/2,0}$	$\frac{1}{2}\left(1+\frac{1}{2}y_0(l)+y_\theta(l)\right)$
$\sin(\sqrt{2}p\theta/2)$	$p/2$	-1	PBC	$x_{p/2,0}$	$\frac{1}{2}\left(1+\frac{1}{2}y_0(l)-y_\theta(l)\right)$
Marginal-like	0	1	PBC	x_{marg}	$2-y_0(l)\left(1+\frac{4}{3}t\right)$
$\cos(\sqrt{2}p\theta)$-like	$p=0$	1	PBC	$x_{p,0}$	$2+2y_0(l)\left(1+\frac{2}{3}t\right)$
$\sin(\sqrt{2}p\theta)$	$p=0$	-1	PBC	$x_{p,0}$	$2+y_0(l)$
$\cos(\sqrt{2}\phi/2)$	0	1	TBC	$x_{0,\frac{1}{2}}$	$\frac{p^2}{32}\left(1-\frac{1}{2}y_0(l)\right)$
$\sin(\sqrt{2}\phi/2)$	0	-1	TBC	$x_{0,\frac{1}{2}}$	

case, the renormalized scaling dimensions have been given in [7]. On the other hand, in the low-temperature limit $\lambda \to \infty$, the model has a p-fold degenerated ground state. In this case, the simple $K = 4$ version of the level spectroscopy is no longer available. The BKT critical transition point can be determined by the $K = p^2/4$ version of the level spectroscopy, which we will explain in subsection 3.2 (see also [6, 9]). The renormalized scaling dimensions near $K = p^2/4$ are shown in table 1 (note that implementing TBC is the same as introducing a half-charge into the effective model [12]). Assuming conformal invariance, the scaling dimensions $x_{m,n}$ are related to the energy gaps of the finite size system with periodic boundary conditions as

$$x_{m,n} = \frac{L}{2\pi v}(E_{m,n}(L) - E_{\mathrm{g}}(L)), \tag{29}$$

where L is the system size, $E_{\mathrm{g}}(L)$ is the ground state energy and v is the velocity of the system [14]. And the central charge c, which we will use to confirm the universality class of the system, is given by the finite size correction of the system as

$$E_{\mathrm{g}}(L) = e_{\mathrm{g}}L - \frac{\pi v c}{6L}, \tag{30}$$

where e_{g} is the bulk ground state energy per site [15].

At last, we discuss the dual transformation in this dual sine-Gordon model, corresponding to the discrete cases (11) and (14). When we apply the dual transformation

$$\phi \to p\theta, \qquad p\theta \to \phi \tag{31}$$

to the dual sine-Gordon model (19), we get the dual form

$$\mathcal{L}_{\mathrm{dual}} = \frac{p^2}{2\pi K}(\nabla\theta)^2 + \frac{y_\phi}{2\pi\alpha^2}\cos p\sqrt{2}\theta + \frac{y_\theta}{2\pi\alpha^2}\cos\sqrt{2}\phi. \tag{32}$$

When we substitute (22) into (32), we obtain the same form as (19) aside the coefficients; setting $y_\phi = y_\theta$ and $p = K$, we get the same equation as (19) including the coefficients. This is the self-dual point of the dual sine-Gordon model. Note that with the dual transformation (31), fields in table 1 are exchanged as $\cos\sqrt{2}p\theta/2 \leftrightarrow \cos\sqrt{2}\phi/2$ etc. These exchanges mean the degeneracy of the excitation spectra at the self-dual point.

The quantum discrete model has the self-dual point (14), and now this self-duality is extended to the continuum model. In the following section, we numerically estimate the self-dual point with good precision.

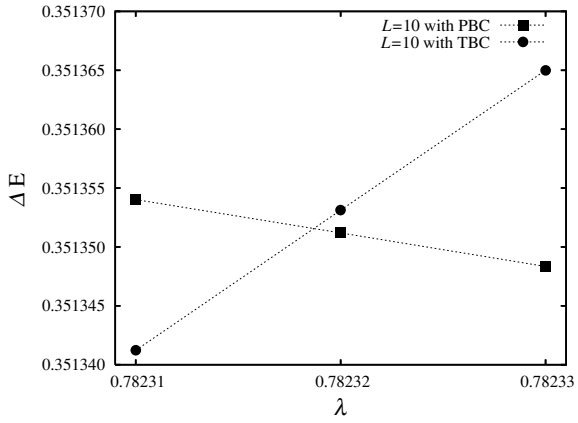

Figure 1. The crossing point for $L = 10$ in lower λ.

3. Numerical results

In the previous section we have argued the general p-state clock model. In this section we show the numerical results for the $p = 6$ case. The energy spectra labelled by Q and $\tilde{Q} = 0, p/2$ are obtained by the Lanczos method. The system size is up to $L = 10$.

First, we determine the two BKT transition points. These points have essentially a different property, because one is located between the non-degenerate ground state phase and the BKT critical phase, while the other is located between the sixfold degenerate ground states phase and the BKT critical phase.

Next, to confirm the universality class of the BKT critical phase, we calculate the ratio of the scaling dimensions, the central charge c and the parameter K of the sine-Gordon model. Additionally, we discuss a level crossing at the self-dual point.

3.1. The BKT transition point in lower λ

The BKT transition point between the non-degenerate ground state phase and the BKT critical phase is given by the level crossing of the following energy gaps [7]:

$$x_{2,0} = \frac{L}{2\pi v}\Delta E(Q = 2, \mathrm{PBC}), \tag{33}$$

$$x_{0,\frac{1}{2}} = \frac{L}{2\pi v}\Delta E(Q = 0, \mathrm{TBC}, C = 1). \tag{34}$$

ΔE is an energy gap between the ground state and the excitation state labelled by Q and $\tilde{Q} = 0, p/2$. A periodic boundary condition (PBC) corresponds to $\tilde{Q} = 0$, and a twisted boundary condition (TBC) corresponds to $\tilde{Q} = p/2$. The ground state has $Q = 0, C = 1$ with PBC.

Figure 1 shows the crossing point for $L = 10$. On the BKT transition point, we can eliminate logarithmic corrections including higher terms by the level spectroscopy, but still we have finite size corrections due to the irrelevant field $L_{-2}\bar{L}_{-2}\mathbf{1}$ ($x = 4$) [14] which behaves in the order of $1/L^2$, $1/L^4$, etc. In figure 2, we show the extrapolation of the level crossing as L tends to infinity. The result is $\lambda = 0.781\,83$.

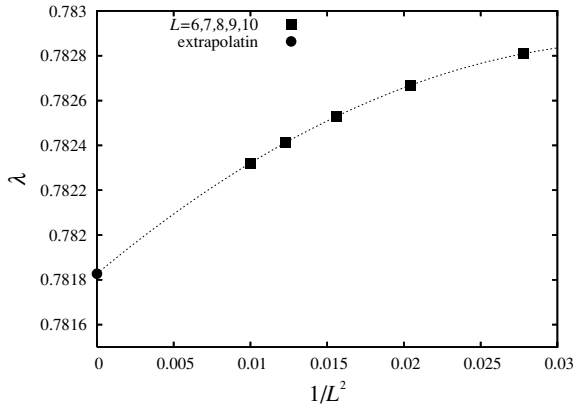

Figure 2. The lower BKT transition point. The extrapolated value is $\lambda = 0.781\,83$.

3.2. The BKT transition point in upper λ

From table 1, on the BKT critical line $y_\theta(l) = -y_0(l)$, we find $x_{p/2,0} : x_{0,1/2} = p^2/32 : 1/2$ including the $1/\ln L$ correction. Therefore from equation (29), the crossing point

$$\frac{p^2}{16}\frac{L}{2\pi v}\Delta E(Q = p/2, \text{PBC}, C = 1) = \frac{L}{2\pi v}\Delta E(Q = 0, \text{TBC}, C = 1) \quad (35)$$

gives the BKT critical point. Simultaneously, the logarithmic corrections are eliminated. This is the advantage of the level spectroscopy method.

From above, for $p = 6$, the upper BKT critical point between the sixfold degenerate ground state phase and the critical phase is given by the crossing point of the following scaling dimensions [6, 9]:

$$\frac{9}{4}x_{3,0} = \frac{9}{4}\frac{L}{2\pi v}\Delta E(Q = 3, \text{PBC}, C = 1), \quad (36)$$

$$x_{0,\frac{1}{2}} = \frac{L}{2\pi v}\Delta E(Q = 0, \text{TBC}, C = 1). \quad (37)$$

For $Q = 0, 3$, the charge conjugation C can be a good quantum number as mentioned at the end of subsection 2.2.

The crossing point is shown in figure 3 for $L = 10$. This crossing eliminates the logarithmic correction order of $\mathcal{O}(1/\ln L)$. But still higher order corrections like $O(1/(\ln L)^2)$ may remain in this case. Anyway we extrapolate the BKT transition point for the large limit of the system size L as in figure 4. We obtain $\lambda = 1.2851$.

The product of λ_{upper} and λ_{lower} can be used to check the consistency of the self-duality; that is, $\lambda_{\text{upper}} \cdot \lambda_{\text{lower}}$ should be unity. Actually, from the numerical data, we get

$$\lambda_{\text{upper}} \cdot \lambda_{\text{lower}} = 1.0047. \quad (38)$$

3.3. Universality class

3.3.1. Duality. We numerically calculate the change of K via λ by the ratio of scaling dimensions:

$$\frac{\Delta E(Q = 3, \text{PBC}, C = 1)}{\Delta E(Q = 0, \text{TBC}, C = 1)} = \frac{x_{3,0}}{x_{0,\frac{1}{2}}} = 36/K^2. \quad (39)$$

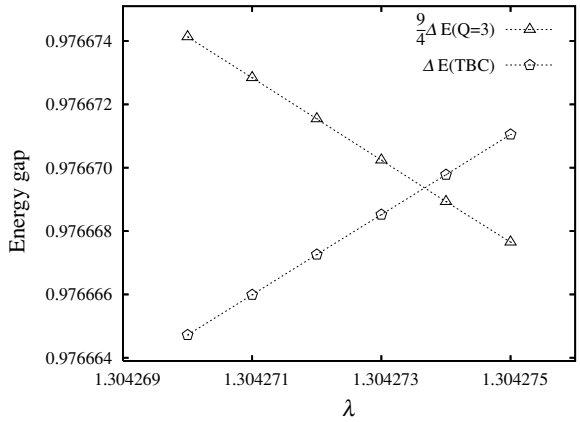

Figure 3. The crossing point for $L = 10$ in upper λ.

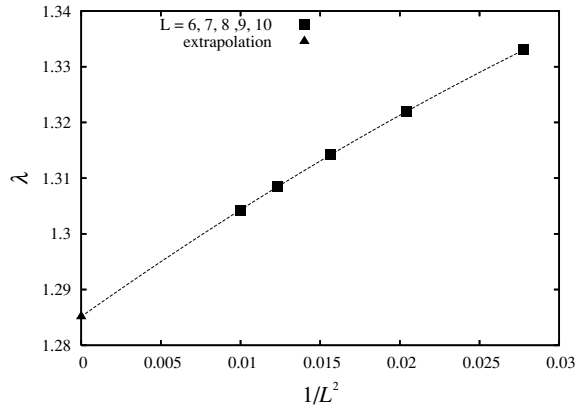

Figure 4. The upper BKT transition point. The extrapolated value is $\lambda = 1.2851$.

This ratio may have the logarithmic correction $\mathcal{O}(1/(\ln L)^2)$ in the vicinity of the BKT transition points. But at the self-dual point, $\Delta E(Q = 3, \text{PBC}, C = 1)$ and $\Delta E(Q = 0, \text{TBC}, C = 1)$ exactly degenerate; therefore $K = 6$. In fact, in figure 5, the excitations $\Delta E(Q = 3, \text{PBC}, C = 1)$ and $\Delta E(Q = 0, \text{TBC}, C = 1)$ precisely cross at $\lambda = 1$. This is because at the self-dual point the dual sine-Gordon model has the higher symmetry discussed in (31) and (32).

This crossing at $\lambda = 1$ is consistent with the duality in the discrete model,

$$H_Q^{\tilde{Q}}(\lambda) = \lambda H_{\tilde{Q}}^{Q}(1/\lambda), \tag{40}$$

where $H_Q^{\tilde{Q}}$ is a sector Hamiltonian labelled by Q and $\tilde{Q}$ [4, 16].

We emphasize that our novel point is not the degeneracy but the connection of the discrete Z_p model and the Z_p dual sine-Gordon model.

Even though when one studies another lattice model which does not have self-duality explicitly, but if it could be mapped to the dual sine-Gordon model, one can check the self-dual point of the dual sine-Gordon model through this level crossing. For example, such an approach will be useful in the vicinity of the multicritical point studied by Otsuka *et al* [9].

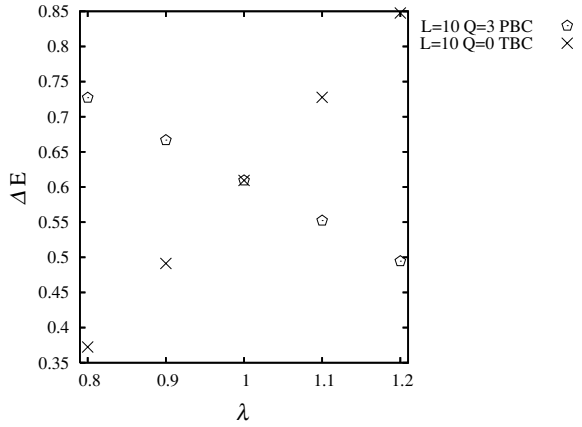

Figure 5. Level crossing occurs exactly at the self-dual point $\lambda = 1$. These levels are $\Delta E(Q = 3, \mathrm{PBC}, C = 1)$ and $\Delta E(Q = 0, \mathrm{TBC}, C = 1)$ for $L = 10$. The other system sizes also show the exact crossing.

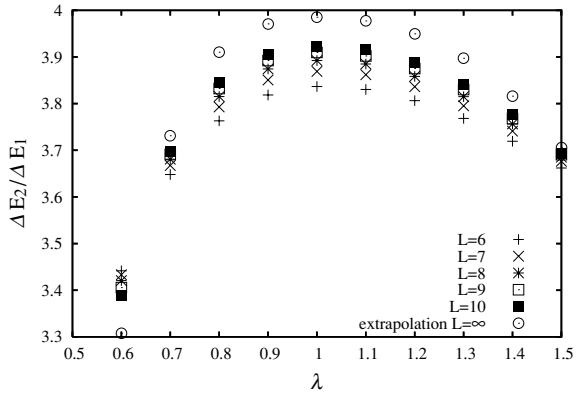

Figure 6. Ratio of scaling dimensions $x_{2,0}/x_{1,0}$. In the BKT critical region the ratio $x_{2,0}/x_{1,0} \simeq 4$.

3.3.2. Ratio of scaling dimensions. The ratio of scaling dimensions

$$\frac{x_{2,0}}{x_{1,0}} = \frac{\Delta E(Q = 2, \mathrm{PBC})}{\Delta E(Q = 1, \mathrm{PBC})} = 4 + \mathcal{O}(L^{-2}) + \text{higher order terms} \tag{41}$$

is also useful to check the universality class of the BKT critical region. The finite correction behaves in the order of $\mathcal{O}(L^{-2})$. Figure 6 shows this ratio.

3.3.3. Central charge. Generally, in a BKT critical region a renormalization equation flows to the Gaussian fixed line. Therefore the universality class of the BKT critical region agrees with the Gaussian model. The Gaussian model is known to have the central charge $c = 1$, so the BKT critical region is also characterized by $c = 1$ [17].

From the conformal field theory, the central charge is related to the ground state of the finite system as [18]

$$E_{\mathrm{g}}(L) = e_{\mathrm{g}} L - \frac{\pi v c}{6L}\left(1 + \mathcal{O}\left(\frac{1}{(\ln L)^3}\right)\right), \tag{42}$$

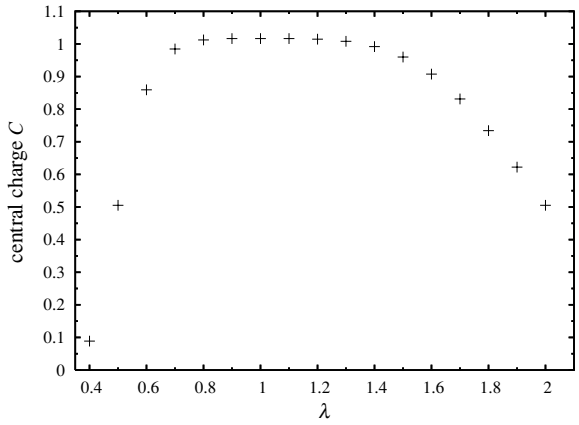

Figure 7. Effective central charge extrapolated from the finite system's results.

where e_g is a free energy per site and v is a velocity of the system. In this case the logarithmic correction is small enough, so the central charge is numerically a good index of universality class.

We calculate the effective central charge for the system size up to $L = 10$ and extrapolated to $L \to \infty$. In figure 7, we show the extrapolated values of the central charge. This result is consistent with the two transition points which we have determined.

4. Conclusion

We have extended the level spectroscopy to determine the BKT critical point between the multi-fold degenerated phase and the critical region. For a physical application we studied the 1D quantum p-clock model which has the Z_p symmetry. The numerical calculations were performed for $p = 6$.

Also we discussed about the self-duality of the dual sine-Gordon model. The self-dual point has been numerically determined without the logarithmic corrections. While this is trivial in the clock model because the level crossing of $\Delta E(Q = p/2, \text{PBC})$ and $\Delta E(Q = 0, \text{TBC})$ is the exact result from the self-duality at $\lambda = 1$, but this result also means that there is no correction term in the language of the dual sine-Gordon model (19). On the other hand, the BKT transition points that we have determined may have some correction, because we have ignored irrelevant terms to derive (25) and (26) from (19). The self-duality approach will be useful for other models which need highly accurate calculations, for instance, the cross-over near the multicritical point dealt in [9].

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